



Multiple Input Production Economics for Farm Management

AAE 320

Paul Mitchell

pdmitchell@wisc.edu 608-320-1162



Learning Goals

- Economics of identifying optimal input use when have two inputs to choose at the same time
 - Multiple Input Production Function
- Both in tabular form and using calculus
 - How to use the new optimality condition
- Key concept: Tradeoff or Substitution between inputs

Multiple Input Production

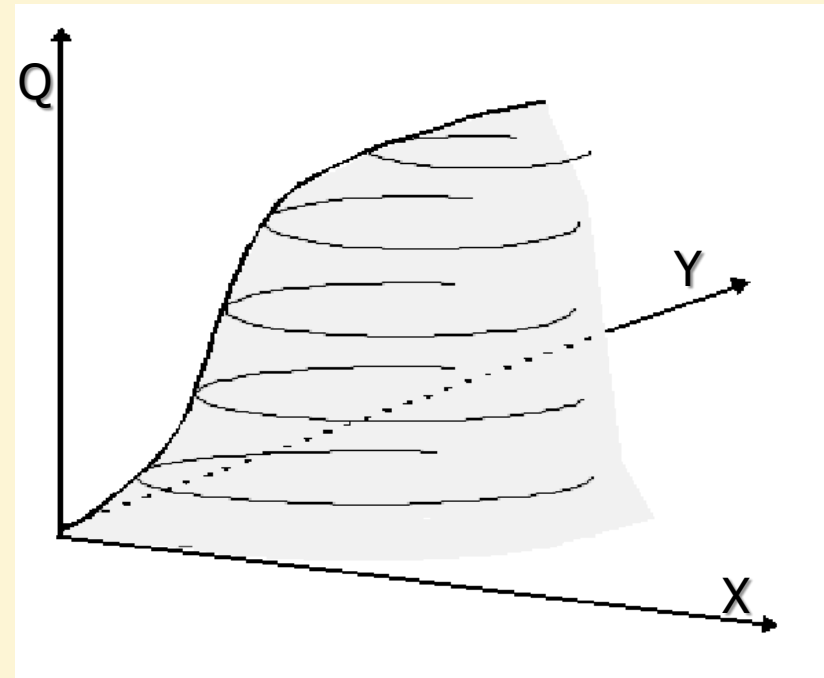
- Most agricultural production processes have more than one input, e.g., capital, labor, land, machinery, fertilizer, herbicides, insecticides, tillage, water, etc.
- How do you decide how much of each input to use when you are choosing more than one?
- Will derive the Equal Margin Principle and show its use to answer this question
- Will derive it using calculus, so you see where it comes from
- Applies whether you use a function or not, so can apply Equal Margin Principle to the tabular form of the multiple input production schedule
- Multivariate calculus requires use of Partial Derivatives, so let's review

Partial Derivatives

- Derivative of function that has more than one variable
- What's the derivative of $Q = f(x,y)$? Depends on which variable you are talking about.
- Remember a derivative is the slope
- Derivative of $Q = f(x,y)$ with respect to x is the slope of the function in the x direction
- Derivative of $Q = f(x,y)$ with respect to y is the slope of the function in the y direction

Partial Derivatives

- Think of a hill; its elevation Q is a function of the location in latitude (x) and in longitude (y): $Q = f(x,y)$
- At any spot on the hill, defined by a latitude-longitude pair (x,y) , the hill will have a slope in the x direction and in the y direction
- Slope in x direction: $dQ/dx = f_x(x,y)$
- Slope in y direction: $dQ/dy = f_y(x,y)$



Partial Derivatives

- Notation: if have $Q = f(x,y)$
- First Partial Derivatives
 - $dQ/dx = f_x(x,y)$ and $dQ/dy = f_y(x,y)$
- Second (Own) Partial Derivative
 - $d^2Q/dx^2 = f_{xx}(x,y)$ and $d^2Q/dy^2 = f_{yy}(x,y)$
- Second Cross Partial Derivative
 - $d^2Q/dxdy = f_{xy}(x,y)$

Partial Derivatives

- Partial derivatives are the same as regular derivatives, just treat the other variables as constants
- $Q = f(x,y) = 2 + 3x + 6y - 2x^2 - 3y^2 - 5xy$
- When taking the derivative with respect to x , treat y as a constant and vice versa
- $Q = f(x,y) = 2 + 3x + 6y - 2x^2 - 3y^2 - 5xy$
- $f_x(x,y) = 3 - 4x - 5y$
- $Q = f(x,y) = 2 + 3x + 6y - 2x^2 - 3y^2 - 5xy$
- $f_y(x,y) = 6 - 6y - 5x$

Think Break #6

- Give the 1st and 2nd derivatives [$f_x(x,y)$, $f_y(x,y)$, $f_{xx}(x,y)$, $f_{yy}(x,y)$, $f_{xy}(x,y)$] of each function:
- $f(x,y) = 7 + 5x + 2y - 5x^2 - 4y^2 - 11xy$
- $f(x,y) = -5 - 2x + y - x^2 - 3y^2 + 2xy$

Equal Margin Principle

- Given production function $Q = f(x,y)$, find (x,y) to maximize $\pi(x,y) = pf(x,y) - r_x x - r_y y - K$
- FOC's: $d\pi/dx = 0$ and $d\pi/dy = 0$ and solve for pair (x,y)
- $d\pi/dx = pf_x(x,y) - r_x = 0$
- $d\pi/dy = pf_y(x,y) - r_y = 0$
- Just $p \times MP_x = r_x$ and $p \times MP_y = r_y$
- Just $MP_x = r_x/p$ and $MP_y = r_y/p$
- These still hold, but we have more

} Same as before

Equal Margin Principle

- Profit Maximization again implies
- $p \times MP_x = r_x$ and $p \times MP_y = r_y$
- Note $p \times MP_x = r_x$ depends on y and $p \times MP_y = r_y$ depends on x
- These are two equations and both must be satisfied, so rearrange by making the ratio

$$\frac{\cancel{p}MP_x}{\cancel{p}MP_y} = \frac{r_x}{r_y} \qquad \frac{\cancel{p}MP_x}{r_x} = \frac{\cancel{p}MP_y}{r_y}$$

Equal Margin Principle

Equal Margin Principle: expressed mathematically in two ways

1) $MP_x/r_x = MP_y/r_y$

- Ratio of MP_i/r_i must be equal for all inputs

2) $MP_x/MP_y = r_x/r_y$

- Ratio of MP's must equal input price ratio

Intuition for $MP_x/r_x = MP_y/r_y$ Corn Example

- MP_i is bu of corn from the last lb of N fertilizer (bu/lb)
- r_i is \$ per lb of N fertilizer (\$/lb)
- Units for MP_i/r_i (bu/lb) / (\$/lb) = bu/\$
- MP_i/r_i is bu of corn from the last \$ spent on N fertilizer
- MP_i/r_i is how many bushels of corn you get from the last dollar spent on N fertilizer
- **$MP_x/r_x = MP_y/r_y$ means use each input so that the last dollar spent on each input gives the same extra output**

Intuition for $MP_x/MP_y = r_x/r_y$ Corn Example

- MP_x is bu of corn from last lb of N fertilizer (bu/lb N)
- MP_y is bu of corn from last 1k seeds planted (bu/1k seeds)
- $MP_x/MP_y = (1k \text{ seeds/lbs N})$ is how many seeds/ac need to increase if cut N by 1 lb/ac and want to keep yield the same
- **Ratio of marginal products is the substitution rate between Nitrogen and Seeds in the production process**
- Have not talked about substitution between inputs, but the Ratio of the marginal products is the slope of the tradeoff curve between inputs

Intuition for $MP_x/MP_y = r_x/r_y$ Corn Example

- r_x is \$ per lb of N fertilizer (\$/lb N)
- r_y is \$ per 1k seeds (\$/1k seeds)
- r_x/r_y is $(\$/lb N)/(\$/1k seeds) = 1k seeds/lbs N$, or the **substitution rate between Nitrogen and Seeds in the market**
- $MP_x/MP_y = r_x/r_y$ means use inputs so that the substitution rate between inputs in the production process is the same as the substitution rate between inputs in the market

Marginal Rate of Technical Substitution

- The ratio of marginal products (MP_x/MP_y) is the substitution rate between inputs in the production process
 - Slope of the tradeoff curve
- MP_x/MP_y is called the Marginal Rate of Technical Substitution (MRTS): the input substitution rate at the margin for the production technology
- MRTS: If you cut X by one unit, how much must you increase Y to keep output the same
- Optimality condition $MP_x/MP_y = r_x/r_y$ means set substitution rates equal

Equal Margin Principle Intuition

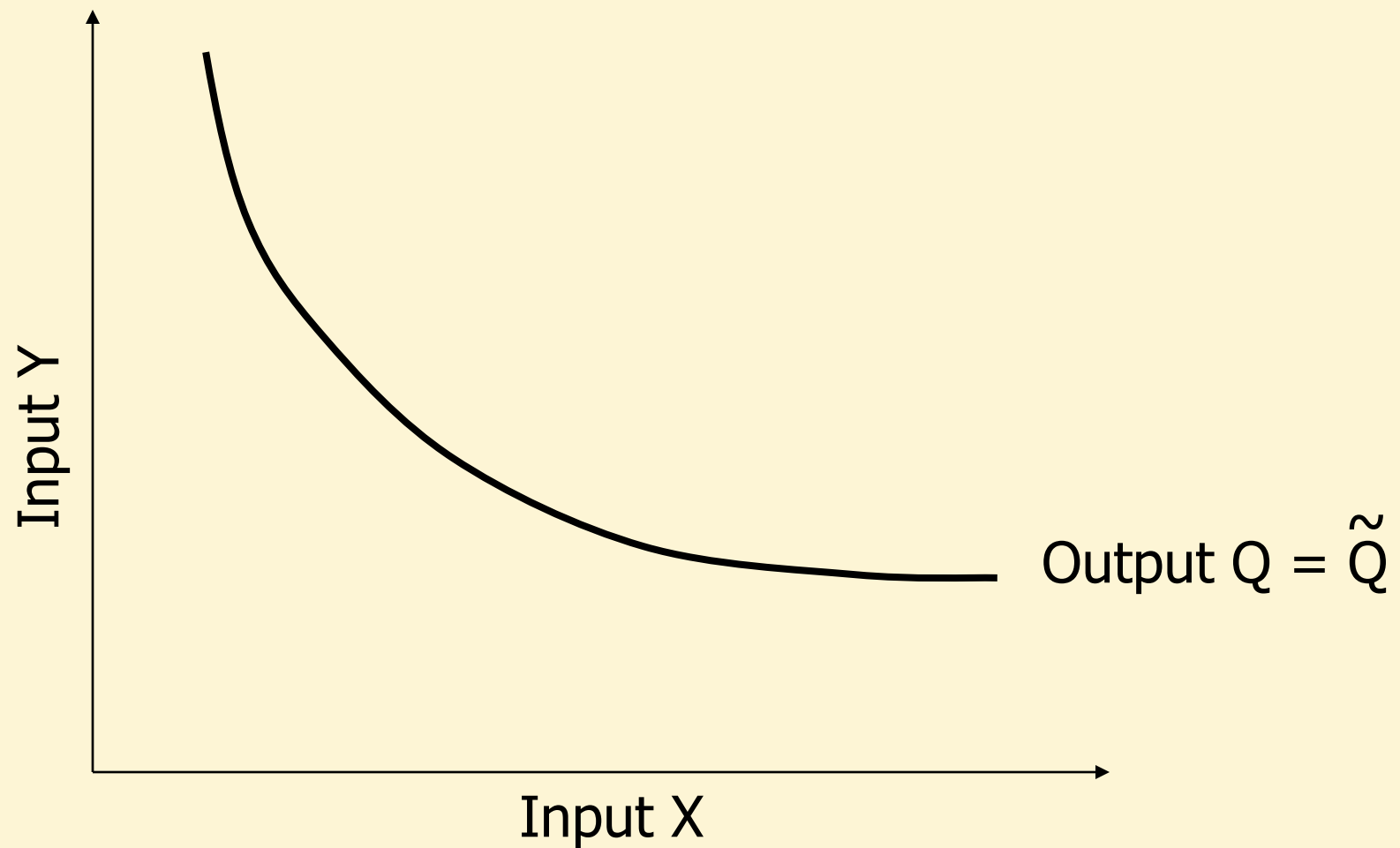
- $MP_x/r_x = MP_y/r_y$ means use inputs so the last dollar spent on each input gives the same extra output at the margin
 - Compare to $p \times MP = r$ or $VMP = r$
- $MP_x/MP_y = r_x/r_y$ means use inputs so the substitution rate at the margin between inputs is the same in the production process as in the marketplace
 - Compare to $MP = r/p$

Equal Margin Principle

Graphical Analysis via Isoquants

- Isoquant (“equal-quantity”) plot or function representing all combinations of two inputs producing the same output quantity
- Intuition: Isoquants are the two dimensional “contour lines” of the three dimensional production “hill”
- First look at in Theory, then a Table


Isoquants in Theory



Swine Feeding Operation: Output (hogs sold/year)

Capital (\$1,000)

Labor (persons/year)

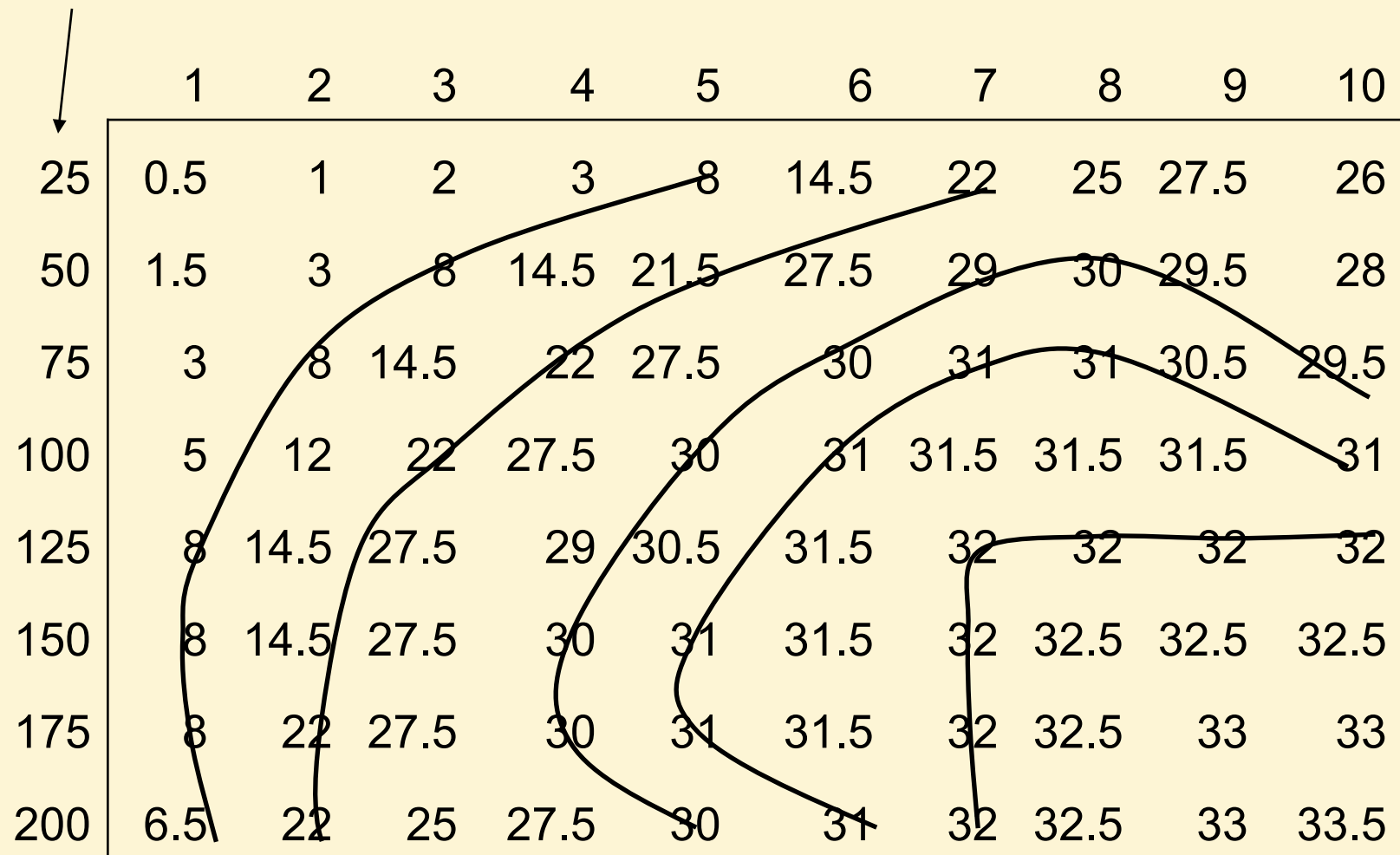


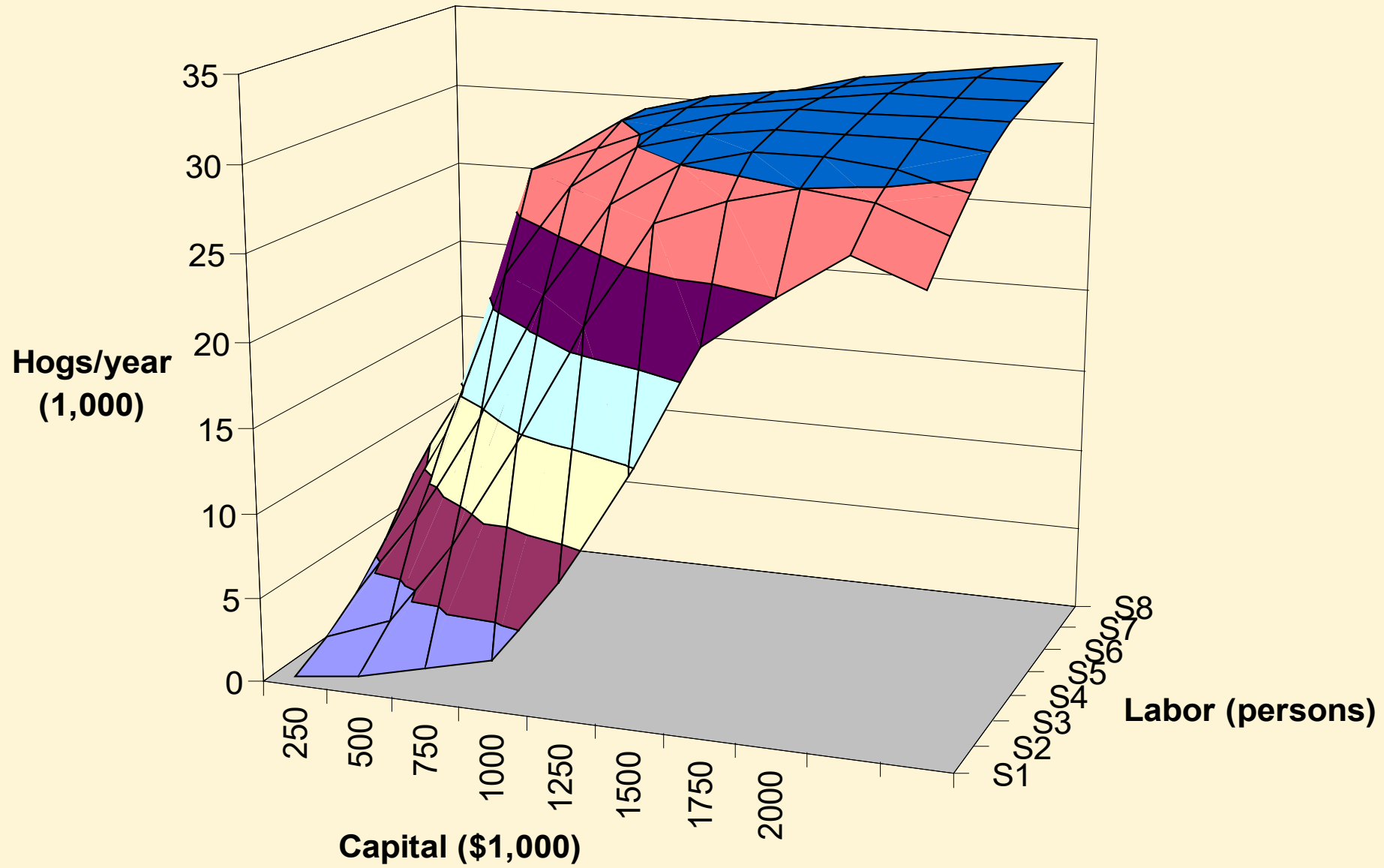
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|-----|------|------|------|------|------|------|------|------|------|
| 25 | 0.5 | 1 | 2 | 3 | 8 | 14.5 | 22 | 25 | 27.5 | 26 |
| 50 | 1.5 | 3 | 8 | 14.5 | 21.5 | 27.5 | 29 | 30 | 29.5 | 28 |
| 75 | 3 | 8 | 14.5 | 22 | 27.5 | 30 | 31 | 31 | 30.5 | 29.5 |
| 100 | 5 | 12 | 22 | 27.5 | 30 | 31 | 31.5 | 31.5 | 31.5 | 31 |
| 125 | 8 | 14.5 | 27.5 | 29 | 30.5 | 31.5 | 32 | 32 | 32 | 32 |
| 150 | 8 | 14.5 | 27.5 | 30 | 31 | 31.5 | 32 | 32.5 | 32.5 | 32.5 |
| 175 | 8 | 22 | 27.5 | 30 | 31 | 31.5 | 32 | 32.5 | 33 | 33 |
| 200 | 6.5 | 22 | 25 | 27.5 | 30 | 31 | 32 | 32.5 | 33 | 33.5 |

Swine feeding operation: Output (hogs sold/year)

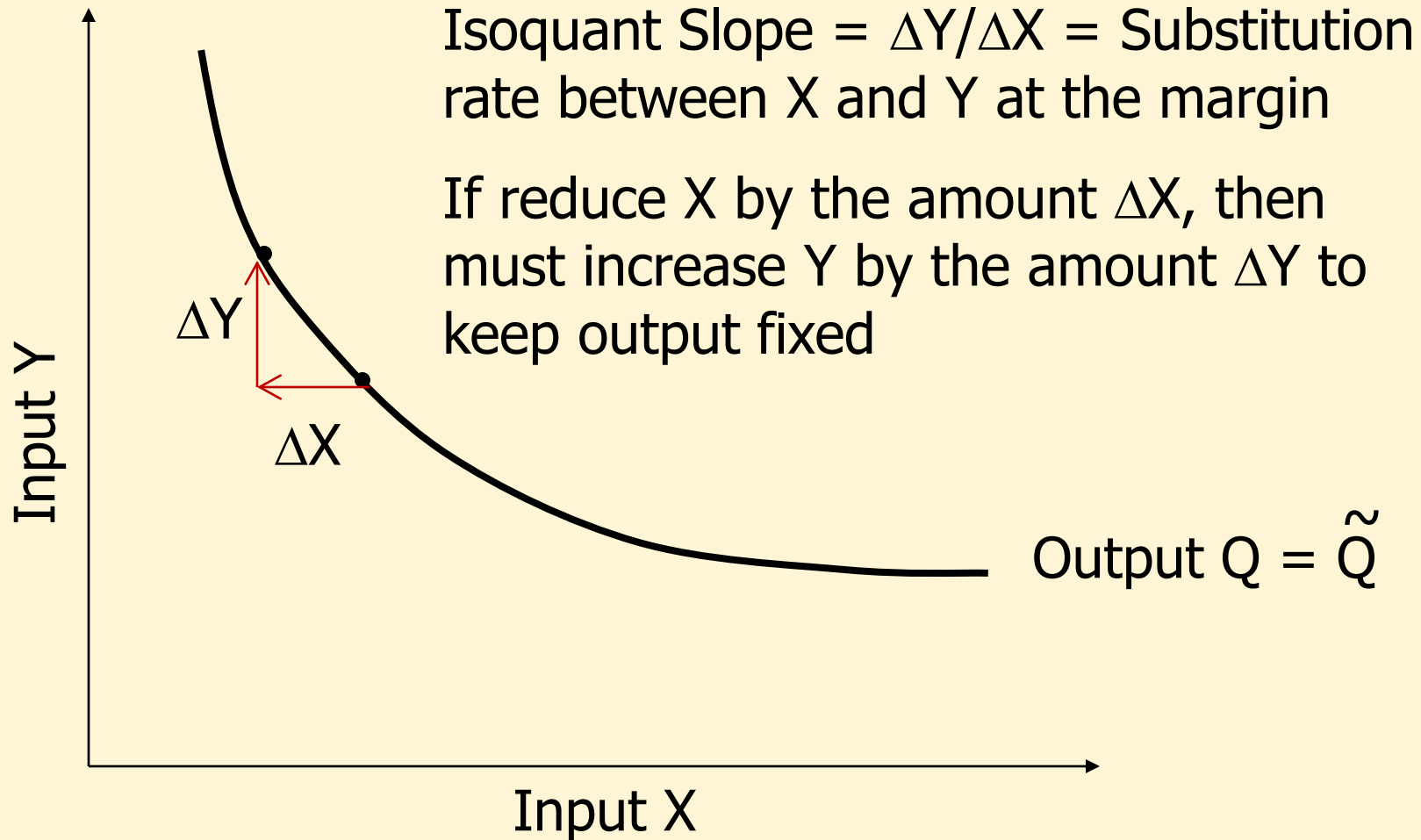
Capital (\$1,000)

Labor (persons/year)

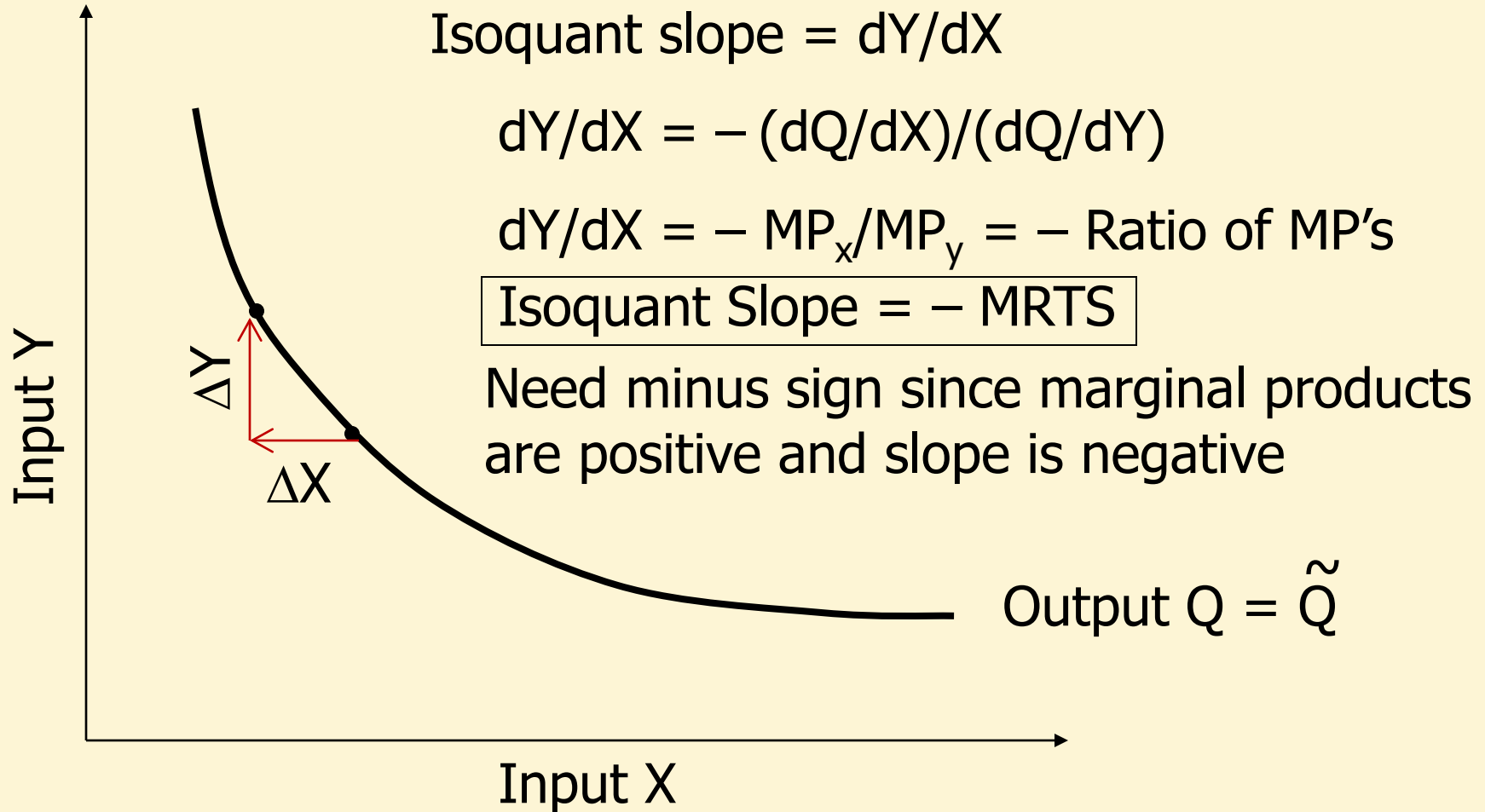




Isoquants and MP_x/MP_y



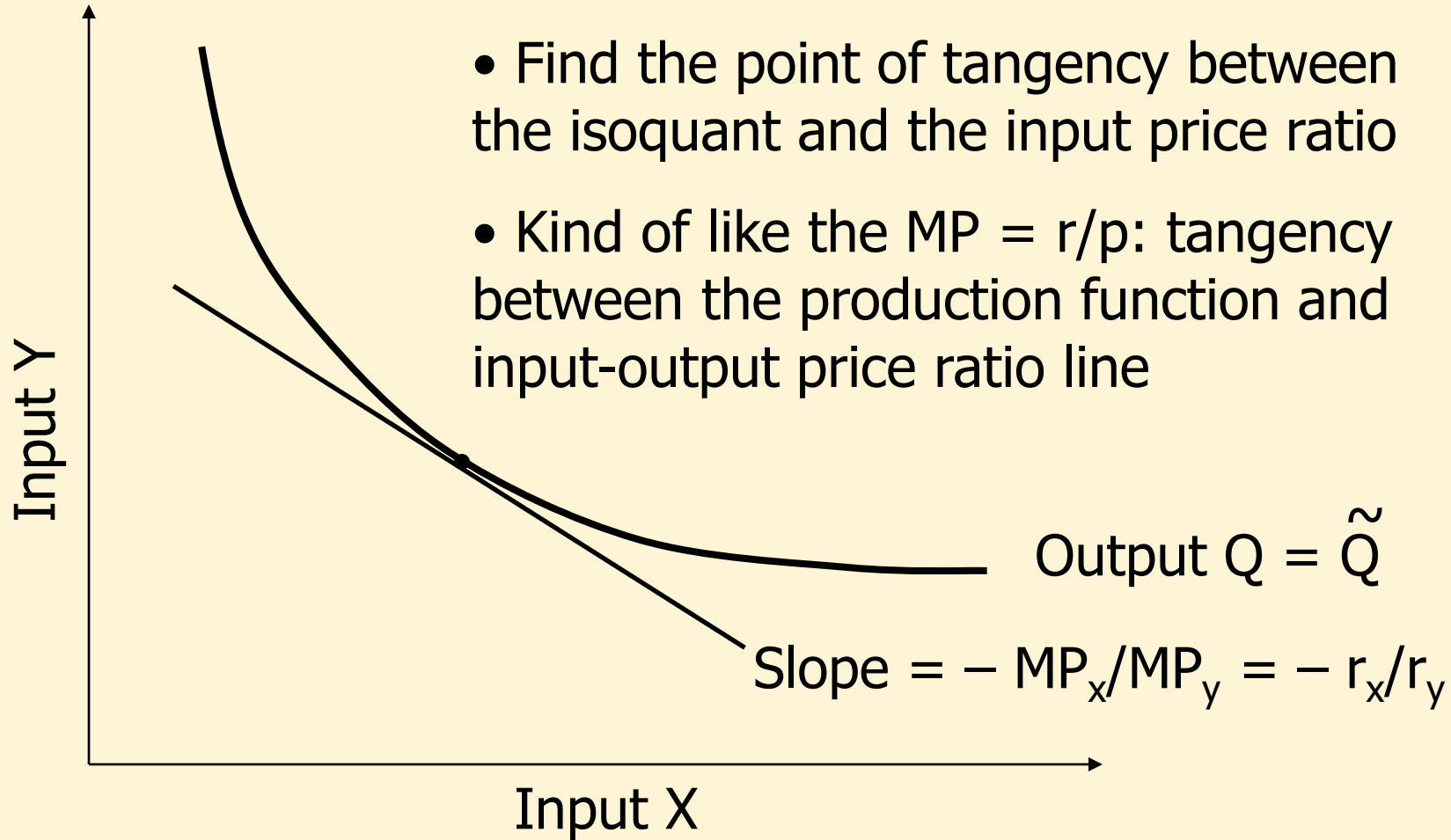
Isoquants and MP_x/MP_y



Isoquants and $MP_x/MP_y = r_x/r_y$

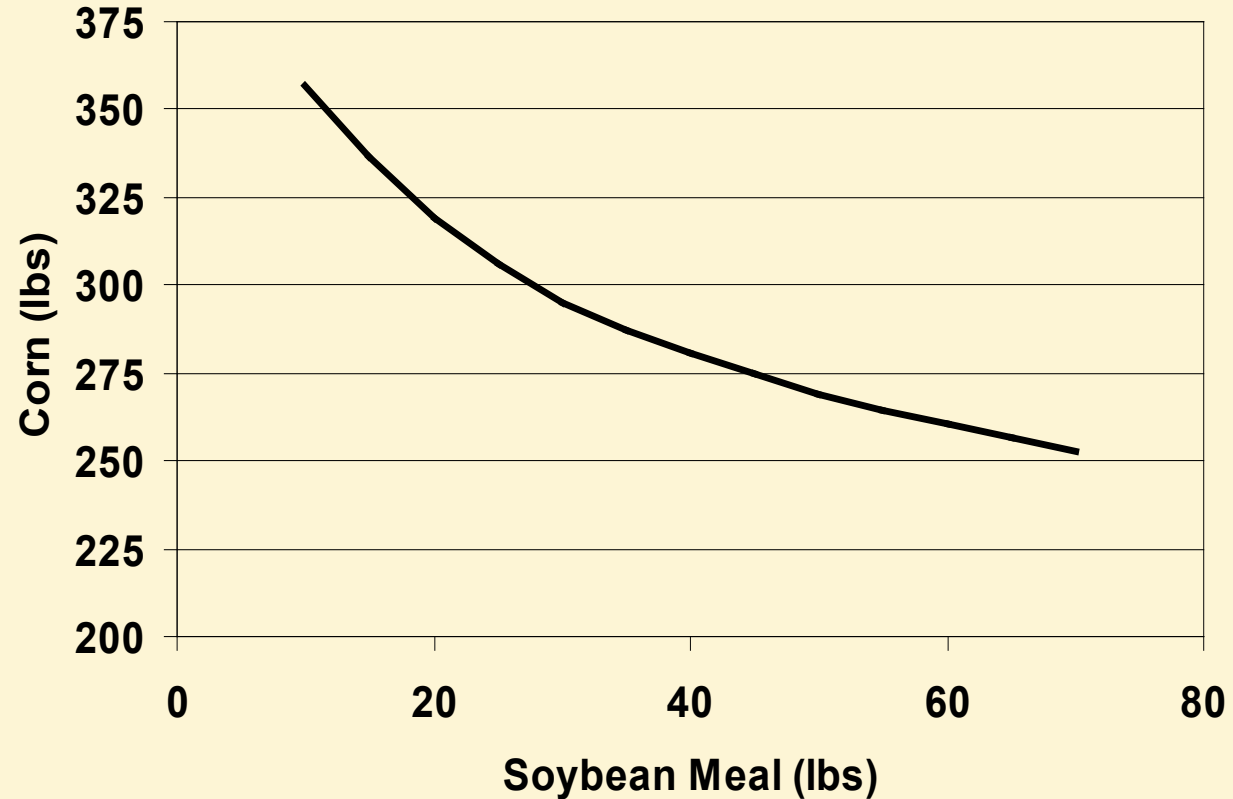
- Isoquant slope ($-MP_x/MP_y$) is (minus) the substitution rate between X and Y at the margin
- Thus define minus the isoquant slope as the “Marginal Rate of Technical Substitution” (MRTS)
- Price ratio r_x/r_y is the substitution rate between X and Y (at the margin) in the marketplace
- Economically optimal use of inputs sets these substitution rates equal, or $MP_x/MP_y = r_x/r_y$

Isoquants and $MP_x/MP_y = r_x/r_y$



| <u>Soybean Meal (lbs)</u> | <u>Corn (lbs)</u> |
|---------------------------|-------------------|
| 10 | 376.8 |
| 15 | 356.3 |
| 20 | 339.0 |
| 25 | 326.0 |
| 30 | 315.0 |
| 35 | 307.5 |
| 40 | 300.6 |
| 45 | 294.6 |
| 50 | 289.2 |
| 55 | 284.4 |
| 60 | 280.2 |
| 65 | 276.4 |
| 70 | 272.9 |

Soybean meal and corn needed for 125 lb feeder pigs to gain 125 lbs



Which feed ration do you use?

| <u>Soyb Meal</u> | <u>Corn</u> | <u>MRTS</u> | MRTS = As increase Soybean Meal, how much can you decrease Corn and keep output constant? |
|------------------|-------------|-------------|---|
| 10 | 376.8 | | |
| 15 | 356.3 | 4.10 | |
| 20 | 339.0 | 3.46 | • Can't use $MRTS = -MP_x/MP_y$ since $\Delta Q = 0$ on an isoquant |
| 25 | 326.0 | 2.60 | Use $MRTS = -\Delta Y/\Delta X$ |
| 30 | 315.0 | 2.20 | |
| 35 | 307.5 | 1.50 | $\Delta X = (SoyM_2 - SoyM_1)$ |
| 40 | 300.6 | 1.38 | $\Delta Y = (Corn_2 - Corn_1)$ |
| 45 | 294.6 | 1.20 | $= - (294.6 - 300.6)/(45 - 40)$ |
| 50 | 289.2 | 1.08 | |
| 55 | 284.4 | 0.96 | $= - (284.4 - 289.2)/(55 - 50)$ |
| 60 | 280.2 | 0.84 | Interpretation: If increase soybean meal 1 lb, decrease corn by 1.20 or 0.96 lbs and keep same gain on hogs |
| 65 | 276.4 | 0.76 | |
| 70 | 272.9 | 0.70 | |

| Soy Meal (lbs) | Corn (lbs) | MRTS | P ratio |
|-------------------|---------------|------|---------|
| 10 | 376.8 | | 2.20 |
| 15 | 356.3 | 4.10 | 2.20 |
| 20 | 339.0 | 3.46 | 2.20 |
| 25 | 326.0 | 2.60 | 2.20 |
| 30 | 315.0 | 2.20 | 2.20 |
| 35 | 307.5 | 1.50 | 2.20 |
| 40 | 300.6 | 1.38 | 2.20 |
| 45 | 294.6 | 1.20 | 2.20 |
| 50 | 289.2 | 1.08 | 2.20 |
| 55 | 284.4 | 0.96 | 2.20 |
| 60 | 280.2 | 0.84 | 2.20 |
| 65 | 276.4 | 0.76 | 2.20 |
| 70 | 272.9 | 0.70 | 2.20 |

Economically Optimal input use is where MRTS = input price ratio, or

$$-\Delta Y/\Delta X = r_x/r_y$$

(Same as $MP_x/MP_y = r_x/r_y$)

Soybean Meal Price
\$176/ton = \$0.088/lb

Corn Price
\$2.24/bu = \$0.04/lb

Ratio: $0.088/0.04 = 2.20$
lbs corn/lbs soy

Keep straight which is X and which is Y!!!

Intuition

- What does $-\Delta Y/\Delta X = r_x/r_y$ mean?
- Ratio $-\Delta Y/\Delta X$ is how much less Y you need if you increase X, or the MRTS
- Cross multiply to get $-\Delta Yr_y = \Delta Xr_x$
- $-\Delta Yr_y =$ cost savings from decreasing Y
- $\Delta Xr_x =$ cost increase from increasing X
- Keep sliding down the isoquant (decreasing Y and increasing X with output staying the same) as long as the cost saving exceeds the cost increase
- Like a partial budget analysis of the cost changes (revenue doesn't change, as output doesn't change)

Main Point on MRTS

- When you have information on inputs needed to generate the same output, you can identify the optimal input combination
- Theory: (MRTS) $MP_x/MP_y = r_x/r_y$
- Practice: (isoquant slope) – $\Delta Y/\Delta X = r_x/r_y$
- **Note how the input in the numerator and denominator changes between theory and practice**
- The numbers in a “table” like I have created actually have underneath them a 2-input production function

Think Break #7

| Grain (lbs) | Hay (lbs) | MRTS | price ratio |
|-------------|-----------|----------------------|----------------------|
| 825 | 1350 | | <input type="text"/> |
| 900 | 1130 | 2.93 | |
| 975 | 935 | <input type="text"/> | |
| 1050 | 770 | 2.20 | |
| 1125 | 625 | 1.93 | |
| 1200 | 525 | <input type="text"/> | |
| 1275 | 445 | 1.07 | |

The table is ratios of grain and hay that put 300 lbs of gain on 900 lb steers.

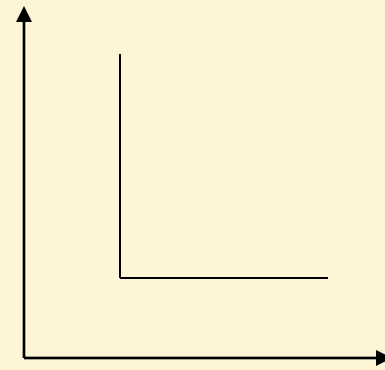
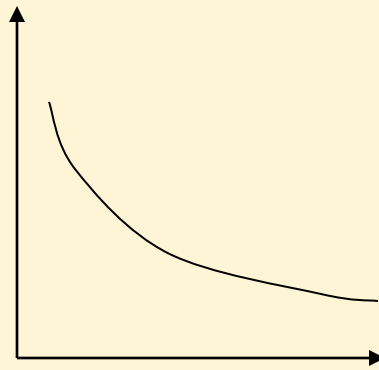
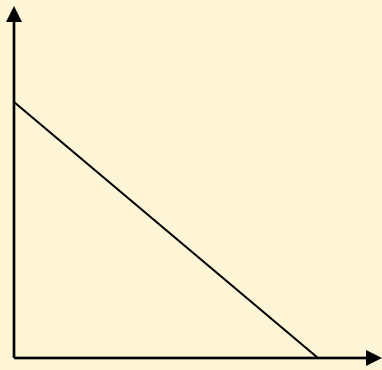
- 1) Fill in the missing MRTS
- 2) If grain is \$0.04/lb and hay is \$0.03/lb, what is the economically optimal feed ration?

Summary

- Can identify economically optimal input combination using tabular data and prices
- Requires input data on the isoquant, i.e., different feed rations that generate the same amount of gain
- Again: we will use calculus to fill in gaps in the tabular form of isoquants

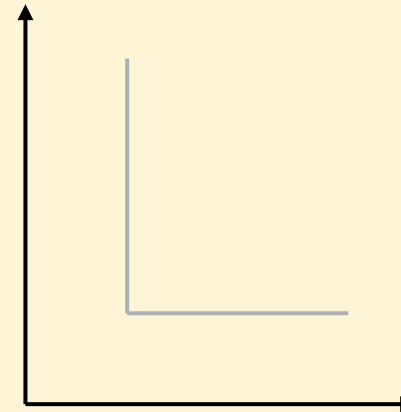
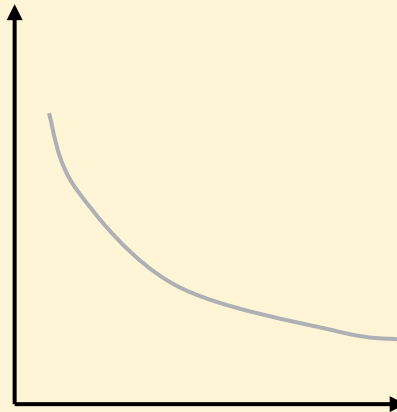
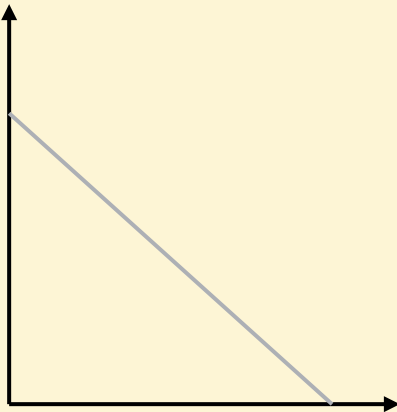
Types of Substitution

- Perfect substitutes: soybean meal & canola meal, corn & sorghum, wheat and barley
- Imperfect substitutes: corn & soybean meal
- Non-substitutes/perfect complements: tractors and drivers, wire and fence posts



Types of Input Substitution

- Perfect substitutes: soybean meal & canola meal, corn & sorghum, wheat and barley
- Imperfect substitutes: corn & soybean meal
- Non-substitutes/perfect complements: tractors and drivers, wire and fence posts



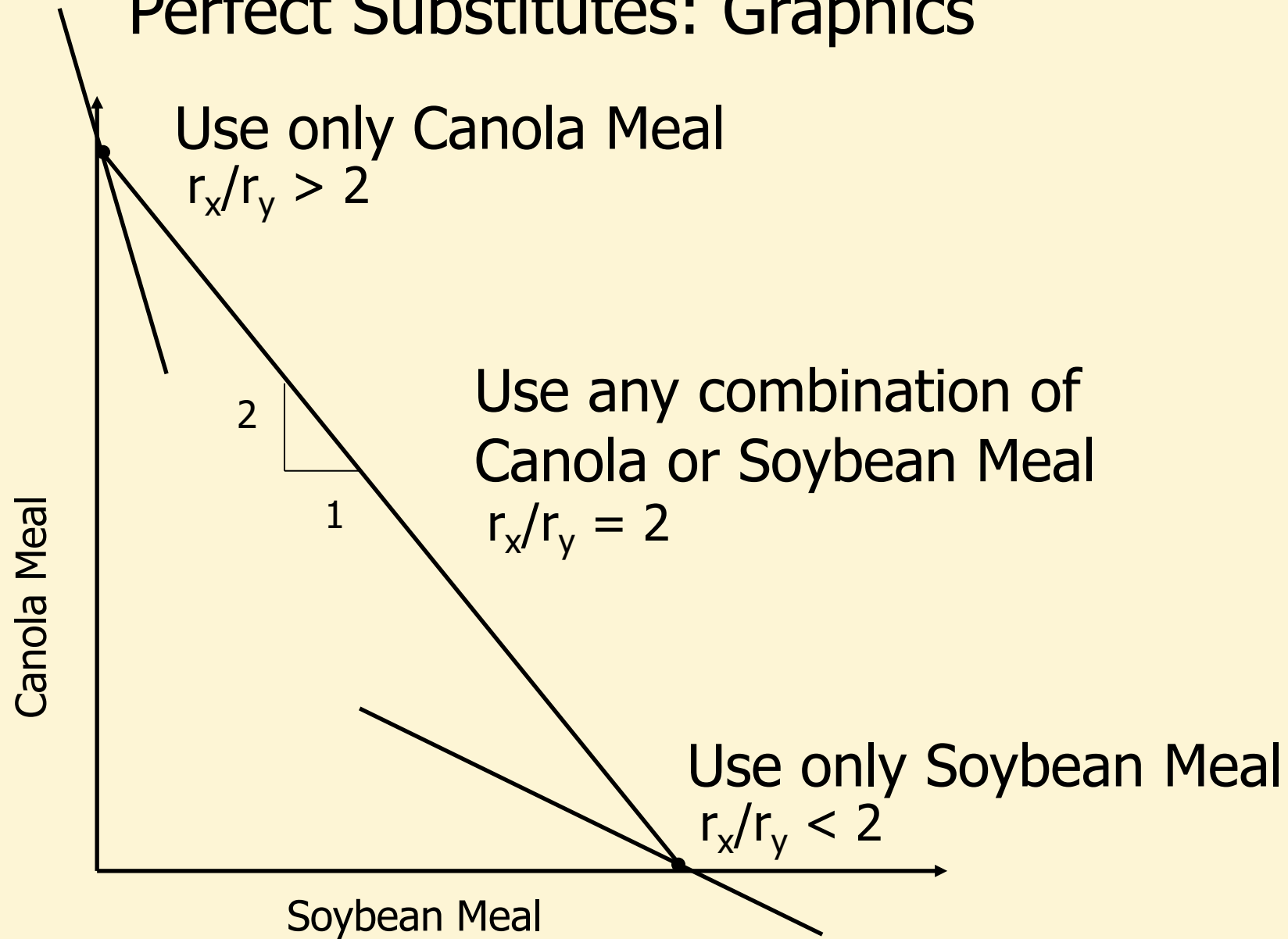
Perfect Substitutes

- MRTS (slope of isoquant) is constant
- Examples: canola meal and soybean meal, or corn and sorghum in a feed ration
- Constant conversion between two inputs
 - 2 lbs of canola meal = 1 lb of soybean meal: $-\Delta\text{Cnla}/\Delta\text{Soyb} = 2$
 - 1.2 bu of sorghum = 1 bu corn
 - $-\Delta\text{Srgm}/\Delta\text{Corn} = 1.2$

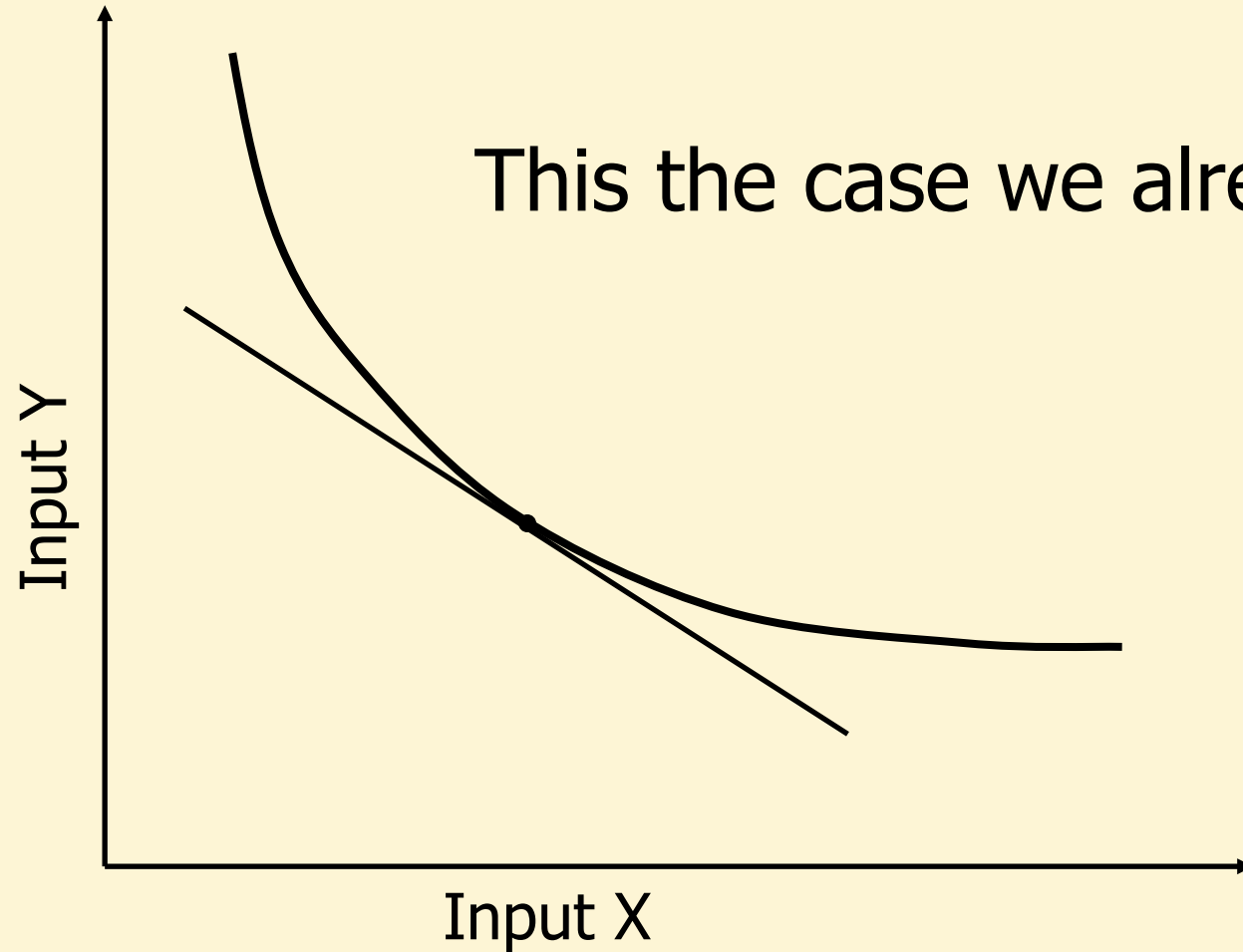
Economics of Perfect Substitutes

- $MRTS = r_x/r_y$ still applies
 $MRTS = k: -\Delta Y/\Delta X = -\Delta Cnla/\Delta Soyb = 2$
- If $r_x/r_y < k$, use only input x
 - soybean meal < twice canola meal price
 - soyb meal \$200/cwt, cnla meal \$150/cwt
- If $r_x/r_y > k$, use only input y
 - soybean meal > twice canola meal price
 - soyb meal \$200/cwt, cnla meal \$90/cwt
- If $r_x/r_y = k$, use any x-y combination

Perfect Substitutes: Graphics



Economics of Imperfect Substitutes



Perfect Complements, or Non-Substitutes

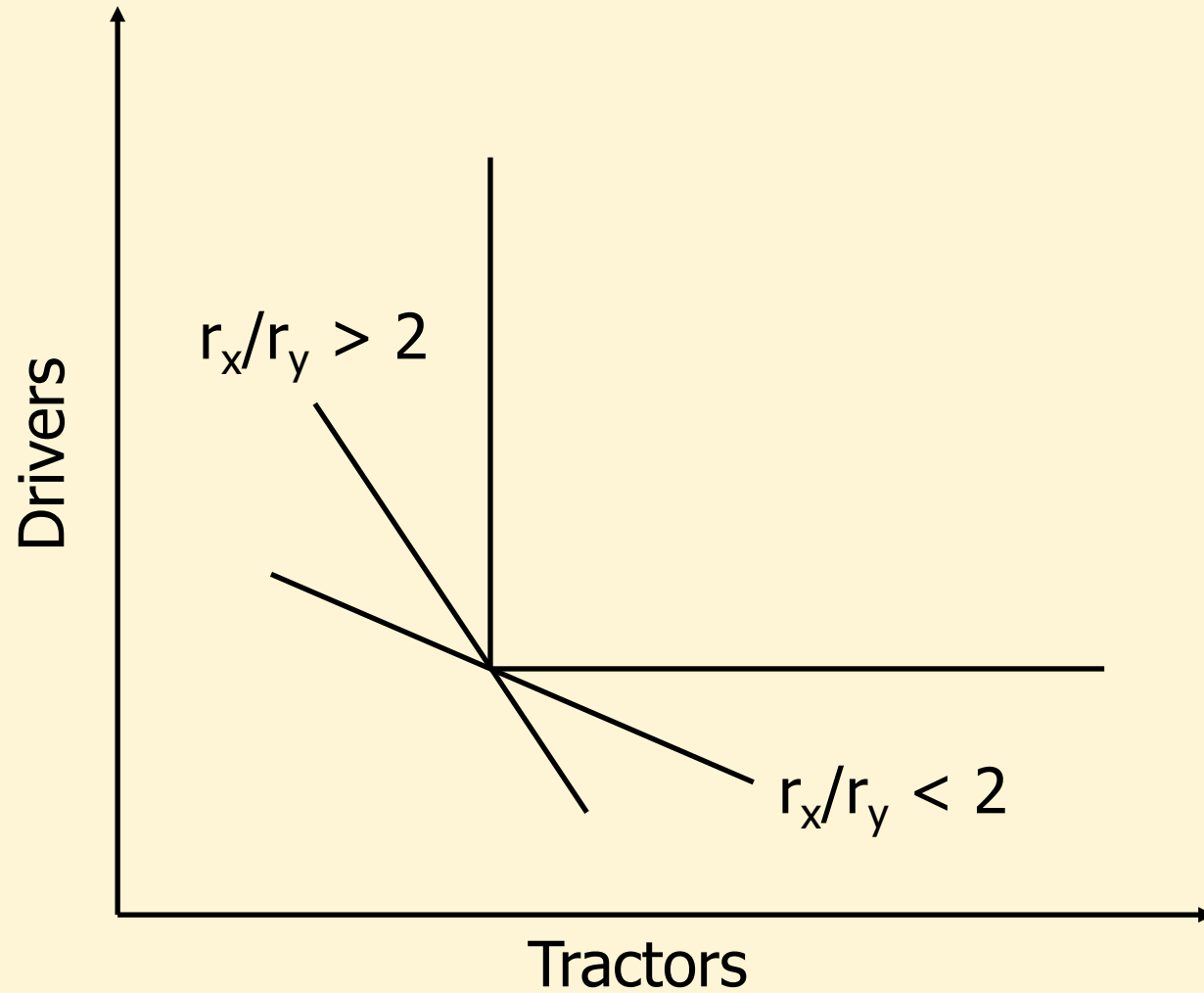
- No substitution is possible
- The two inputs must be used together
- Without the other input, neither input is productive, they must be used together
- Tractors and drivers, fence posts and wire, chemical reactions, digger and shovel
- Inputs used in fixed proportions
1 driver for 1 tractor

Economics of Perfect Complements

- MRTS is undefined, the price ratio r_x/r_y does not identify the optimal combination
- Leontief Production Function
 $Q = \min(aX, bY)$:
- Marginal products of inputs = 0
- Either an output quota or a cost budget, with the fixed input proportions defining how much of the inputs to use

Perfect Complements: Graphics

Price ratio does not matter



Multiple Input Production with Calculus

- Use calculus with production function to find the optimal input combination
- General problem we've seen: Find (x,y) to maximize $\pi(x,y) = pf(x,y) - r_x x - r_y y - K$
- Will get FOC's, one for each choice variable, and SOC's are more complicated
- Economic Problem: How much of the input x and input y do I use to maximize net returns?

Multiple Input Production with Calculus

- If price of output $p = 10$, price of x (r_x) = 2 and price of y (r_y) = 3
- How much x and y do I use to maximize net returns?
- Set up: Max $10(7 + 9x + 8y - 2x^2 - y^2 - 2xy) - 2x - 3y - 8$
- FOC1: $10(9 - 4x - 2y) - 2 = 0$
- FOC2: $10(8 - 2y - 2x) - 3 = 0$
- Find the (x,y) pair that satisfies these two equations
- Same as finding where the two lines from the FOC's intersect

- FOC1: $10(9 - 4x - 2y) - 2 = 0$
- FOC2: $10(8 - 2y - 2x) - 3 = 0$

Solve these two equations for the two unknowns (x, y)

1) Solve FOC1 for x

$$88 - 40x - 20y = 0$$

$$40x = 88 - 20y$$

$$x = (88 - 20y)/40 = 2.2 - 0.5y$$

2) Substitute this x into FOC2 and solve for y

$$77 - 20y - 20x = 0$$

$$77 - 20y - 20(2.2 - 0.5y) = 0$$

$$77 - 20y - 44 + 10y = 0$$

$$33 - 10y = 0, \text{ or } 10y = 33, \text{ or } y = 3.3$$

3) Calculate x:

$$x = 2.2 - 0.5y = 2.2 - 0.5(3.3) = 0.55$$

Substitution Method

- Solve 1st equation for x as function of y
- Substitute this equation into the 2nd equation for x and solve for y
- Substitute this value for y into the 1st equation to calculate x
- Substitute x and y into $f(x,y)$ to get Q and calculate costs and net returns

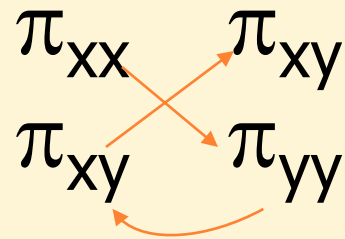
Second Order Conditions

- SOC's are more complicated with multiple inputs, must look at curvature in each direction, plus the “cross” direction (to ensure do not have a saddle point).

1) Own second derivatives must be negative for a maximum:

$$\pi_{xx} < 0, \pi_{yy} < 0 \quad (\text{both } > 0 \text{ for a minimum})$$

2) Another condition: $\pi_{xx}\pi_{yy} - (\pi_{xy})^2 > 0$



- FOC1: $10(9 - 4x - 2y) - 2 = 0$
- FOC2: $10(8 - 2y - 2x) - 3 = 0$
- Second Derivatives
 - 1) $\pi_{xx}: 10(-4) = -40$
 - 2) $\pi_{yy}: 10(-2) = -20$
 - 3) $\pi_{xy}: 10(-2) = -20$

SOC's

- 1) $\pi_{xx} = -40 < 0$ and $\pi_{yy} = -20 < 0$
- 2) $\pi_{xx}\pi_{yy} - (\pi_{xy})^2 > 0$
 $(-40)(-20) - (-20)^2 = 800 - 400 > 0$

Solution $x = 0.55$ and $y = 3.3$ is a maximum

Milk Production

- $M = -25.9 + 2.56G + 1.05H - 0.00505G^2 - 0.00109H^2 - 0.00352GH$
- M = milk production (lbs/week)
- G = grain (lbs/week)
- H = hay (lbs/week)
- Milk \$14/cwt, Grain \$3/cwt, Hay \$1.50/cwt
- What are the profit maximizing inputs?

(Source: Heady and Bhide 1983)

Milk Profit Function

- $\pi(G,H) = 0.14(-25.9 + 2.56G + 1.05H - 0.00505G^2 - 0.00109H^2 - 0.00352GH) - 0.03G - 0.015H$

- FOC's

$$\pi_G = 0.14(2.56 - 0.0101G - 0.00352H) - 0.03 = 0$$

$$\pi_H = 0.14(1.05 - 0.00218H - 0.00352G) - 0.015 = 0$$

- Alternative: jump to optimality conditions

$$MP_G = r_G/p \text{ and } MP_H = r_H/p$$

$$MP_G = 2.56 - 0.0101G - 0.00352H = r_G/p = 0.214$$

$$MP_H = 1.05 - 0.00218H - 0.00352G = r_H/p = 0.107$$

Solve FOC1 for H:

$$2.56 - 0.0101G - 0.00352H = 0.214$$

$$2.346 - 0.0101G = 0.00352H$$

$$H = (2.346 - 0.0101G)/0.00352$$

$$H = 666 - 2.87G$$

Substitute this H into FOC2 and solve for G:

$$1.05 - 0.00218H - 0.00352G = 0.107$$

$$0.943 - 0.00218(666 - 2.87G) - 0.00352G = 0$$

$$0.943 - 1.45 + 0.00626G - 0.00352G = 0$$

$$0.00274G - 0.507 = 0$$

$$G = 0.507/0.00274 = 185 \text{ lbs/week}$$

Substitute this G into the equation for H

$$H = 666 - 2.87G$$

$$H = 666 - 2.87(185)$$

$$H = 666 - 531 = 135 \text{ lbs/week}$$

Check SOC's

$$\pi_{GG} = - (0.14)0.0101 = - 0.001414 < 0$$

$$\pi_{HH} = - (0.14)0.00218 = - 0.000305 < 0$$

$$\pi_{GH} = - (0.14)0.00352 = - 0.000493$$

$$\pi_{GG}\pi_{HH} - (\pi_{GH})^2 = 1.88 \times 10^{-7} > 0$$

SOC's satisfied for a maximum

Calculate Milk Output using this feed ration

$$M = -25.9 + 2.56G + 1.05H - 0.00505G^2 - 0.00109H^2 - 0.00352GH$$

$$M = -25.9 + 2.56(185) + 1.05(135) - 0.00505(185)^2 - 0.00109(135)^2 - 0.00352(185)(135) = 309 \text{ lbs/week}$$

$$309 \times 52 = 16,068 \text{ lbs/year } \textit{(It's 1983!)}$$

Calculate Profit (Net Returns) using this feed ration

$$\text{Profit} = 0.14(309) - 0.03(185) - 0.015(135) = \$41.20/\text{week}$$

Think Break #8

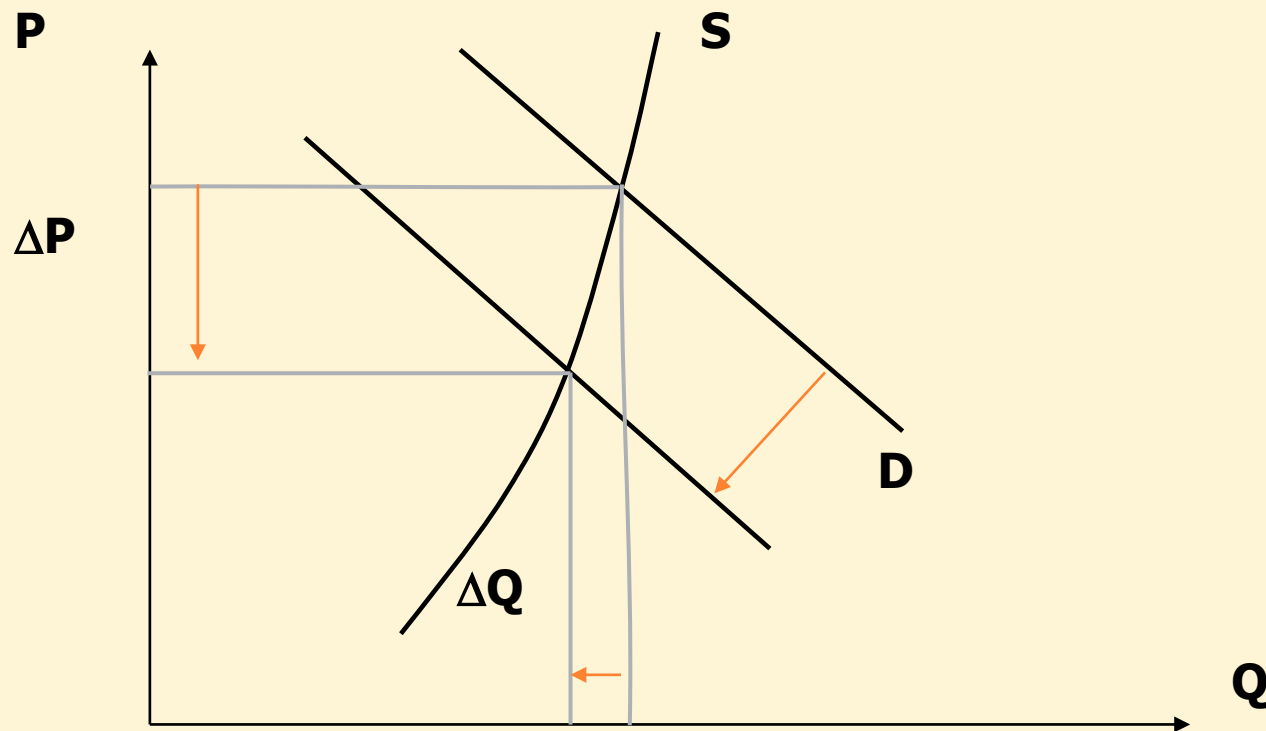
- Repeat the previous problem, but now use a milk price of Milk \$10/cwt
- $M = -25.9 + 2.56G + 1.05H - 0.00505G^2 - 0.00109H^2 - 0.00352GH$
- Prices: Grain \$3/cwt, Hay \$1.50/cwt
- What are the profit maximizing inputs?

Think Break #8 Discussion

- Price decreased 29%, from \$14/cwt to \$10/cwt
- Milk output decreased 0.9%, from 309 to 306 lbs/wk
- Optimal grain use decreased 1.4%, from 185 to 182.5 lbs/wk
- Optimal hay use decreased 12.4%, from 135 to 118.2 lbs/wk
- Net Returns decreased 43%, from \$41.20 to \$23.35/wk
- Economically optimal response to a large price decrease is a small production decrease and a large profit decrease

Think Break #8 Discussion

- The standard agricultural problem: output is non-responsive (inelastic) to price changes, but net returns is responsive



Summary

- Optimality Conditions for multiple inputs have several expressions, all imply Equal Margin Principle
- MRTS, isoquant, and optimal input use are all connected, know the graphics
- Know the different types of input substitution
- Be able to identify optimal input use
 - Tabular form
 - Using calculus