Calculus of Optimization Review for AAE 320
Dr. Paul D. Mitchell (608) 265-6514, pdmitchell@wisc.edu

This document provides a quick review of calculus-based optimization for AAE 320. It assumes a basic understanding of calculus, but that you need a quick "refresher." It goes through the general rules, then works through an example, then gives sample problems (Think Breaks). A useful review of calculus based optimization is available at http://www.swlearning.com/economics/mcguigan/mcguigan9e/web_chapter_a.pdf. This pdf is a web chapter supplement for the text *Managerial Economics* (9th ed.) by McGuigan, Moyer, and Harris. Note that I would like students to know univariate and multivariate unconstrained optimization, and so for AAE 320 you do not need to know the constrained optimization methods described in the web chapter. However, you may want to know them for other AAE courses.

Part 1: Review of Derivatives
Before explaining optimization, first let's review derivatives. A derivative is the slope of a function. For some functions the slope is constant, say 1.5, for other functions, the slope changes depending where on the function you are, and so the slope if a function as well, say $1.5 + 3x$.

Notation for Derivatives
Three different notations are used for 1st and 2nd derivatives of $y = f(x)$

$$1) \quad \frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2} \qquad \text{Newton}$$

2) $f'(x)$ and $f''(x)$ Leibniz
3) $f_x(x)$ and $f_{xx}(x)$ Leibniz

They all mean the same thing, but sometimes one is easier to use than the other. Finally, not that the second derivative is just the derivative of the derivative

Rules of Differentiation (Web Chapter A, Table A.1, page 12)
Constant Function
If $y = f(x) = a$ $f'(x) = 0$

Power Function
If $f(x) = ax^b$ $f'(x) = bax^{b-1}$
If $f(x) = 7x$ $f'(x) = 7$
If $f(x) = 7x^{0.34}$ $f'(x) = 7(0.34)x^{0.34-1} = 2.38x^{-0.66}$

Sum of Functions
If $y = f(x) + g(x)$, then $\dfrac{dy}{dx} = f'(x) + g'(x)$

If $h(x) = 3 + 4x - 7x^2$, then $h'(x) = 4 - 2(7)x^{2-1} = 4 - 12x$
If $k(x) = 56 + 44x^{0.5} - 17x$, then $h'(x) = 0.5(44)x^{0.5-1} - 17 = 22x^{-0.5} - 17$

Review Rules of Differentiation
Find the first and second derivatives of the following functions.

f(x)	f'(x)	f''(x)
$3x^2 + 2x^{0.5}$		
$2x^3 - 5$		
$3\pi x + 5\pi x^2 + 2\pi^2 x^3 + 3e^2$		

Part 2: Unconstrained Optimization (Univariate)
Problem: Choose x to maximize f(x)
1) First Order Condition (FOC): Solve f'(x) = 0 for x to identify x*, which are possible solutions. Note, there may be more than one x* that satisfies the FOC.
2) Second Order Condition (SOC): Evaluate f''(x) at each of the x*'s identified by the FOC to see if the SOC is satisfied. SOC for a maximum is f''(x*) < 0. SOC for a minimum is f''(x*) > 0.
3) If more than x* satisfies both the FOC and the SOC, evaluate the objective function f(x) at each x* to identify the maximum (or minimum).

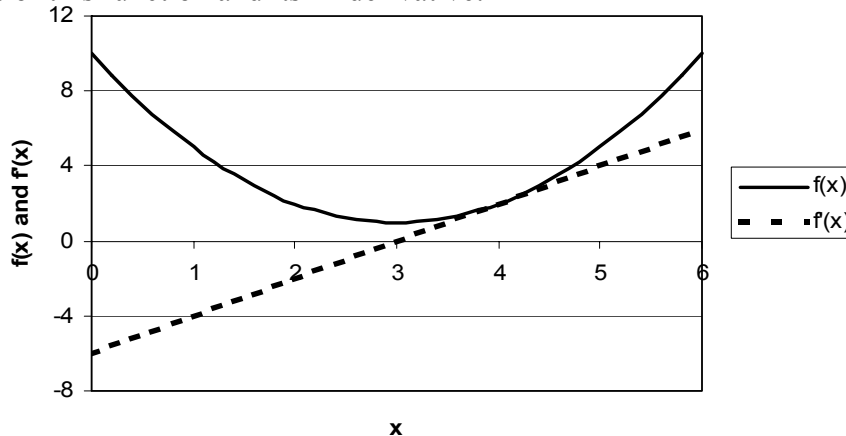If you rely on calculus, ALWAYS check the Second Order Condition—see the stealth bomber example in the Web Chapter A from McGuigan, Moyer, and Harris (http://www.swlearning.com/economics/mcguigan/mcguigan9e/toc.html).

If the calculus confuses you, graph the function. Never underestimate the clarity achieved by a graph.

Example 1
Maximize $f(x) = -10 + 12x - 2x^2$
1. FOC: f'(x0 = 12 − 4x= 0   Satisfied when x = 3
        Is this a maximum or a minimum or an inflection point? How do you know?
2. Check the SOC: f''(x) = − 4 < 0. Negative 2nd derivative, so satisfies SOC for a maximum.

Graphics of this function and its 1st derivative:
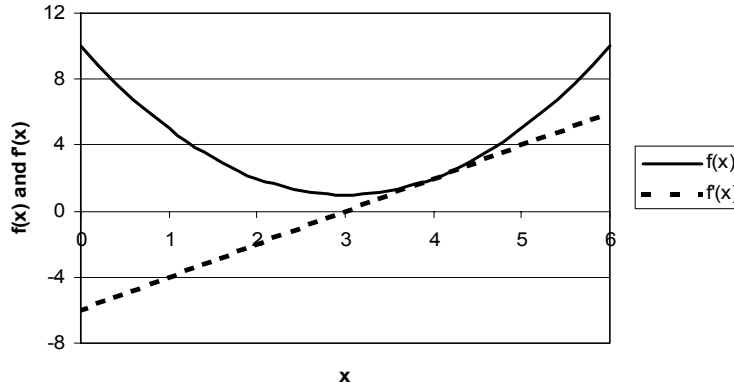
Example 2
Maximize $f(x) = 10 - 6x + x^2$
1. FOC: $f'(x) = -6 + 2x = 0$ Satisfied when $x = 3$

2. Check the SOC: $f''(x) = 2 > 0$. Positive 2nd derivative, so $x = 3$ is not a maximum.
    What is it?

Graphics of this function and its 1st derivative:

What value of x maximizes this function?

Minimization
Minimization is essentially the same as maximization.
If the problem is Minimize h(x), this can be solved as Maximize –h(x)

If you want to learn the different rules for Minimization, they are
1) FOC does not change: Set the first derivative equal to zero and solve for x:
$$h'(x) = 0$$

2) SOC changes sign: Second derivative must be <u>positive</u> for a minimum
$$h''(x) > 0$$

(Look back at Example 2)

Part 3: Review of Partial Derivatives
When the objective function has more than one variable, then the derivative of the function with respect to one of the variable is called a partial derivative. Partial derivatives are essentially the same as regular derivatives, except that the other variables are treated as constants. A few examples will make partial derivatives fairly clear. However, first we need notation.

Notation for partial derivatives
Several forms of notation exist for partial derivatives. The key point to remember is that the notation indicates the variable with respect to which the derivative is taken. For this class, we will sue the following notations (there are others that I will not use).

3

For the function f(x,y), the first partial derivative of $q = f(x,y)$ with respect to x is denoted $f_x(x,y)$, of $f_x(\cdot)$, or simply f_x. Alternatively, I will also use dq/dx. Technically, partial derivatives in this notation are written $\partial q/\partial x$, but I find typing ∂ a pain. Note, ∂ is just another version of the Greek letter δ (delta), but is usually called "partial" when used in this context, so $\partial q/\partial x$ is called "partial q, partial x."

The first partial derivative with respect to y is then $f_y(x,y)$, with all the other notations then following logically. The second partial derivative is simply the partial derivative of the first partial derivative. For notation, the second own partial derivative of f(x,y) with respect to x is $f_{xx}(x,y)$, the second own partial derivative of f(x,y) with respect to y is $f_{yy}(x,y)$. Finally, the (second) cross partial derivative is $f_{xy}(x,y) = f_{yx}(x,y)$, that is the partial derivative of $f_x(x,y)$ with respect to y, or the partial derivative of $f_y(x,y)$ with respect to x (both are the same).

<u>Examples of Partial Derivatives:</u>
Given the function $q(x,y,z) = 3xy + 5x^2y + 2x^3y^2 + 3z$

First Partial Derivatives
$q_x(x,y,z) = 3y + 10xy + 6x^2y^2$
$q_y(x,y,z) = 3x + 5x^2 + 4x^3y$
$q_z(x,y,z) = 3$

Second (Own) Partial Derivatives
$q_{xx}(x,y,z) = 10y + 12xy^2$
$q_{yy}(x,y,z) = 4x^3$
$q_{zz}(x,y,z) = 0$

Second Cross Partial Derivatives
$q_{xy}(x,y,z) = q_{yx}(x,y,z) = 3 + 10x + 12x^2y$
$q_{xz}(x,y,z) = q_{zx}(x,y,z) = 0$
$q_{yz}(x,y,z) = q_{zy}(x,y,z) = 0$

Part 4: Unconstrained Optimization (Multivariate)
Problem: Choose x,y to maximize f(x,y)
1) First Order Conditions (FOCs): Solve the two equations $f_x(x,y) = 0$ and $f_y(x,y) = 0$ for the pair (x,y). Typically, these will be linear equations, and so you are finding the x,y coordinates of where the two lines intersect. For this class, there will be one x,y pair that will satisfy the FOC's.
2) Second Order Conditions (SOCs): There are three conditions, since you must look at the curvature not only in the x and y directions, but also in the cross direction to ensure you do not have a "saddle point." The SOC's for a maximum are $f_{xx} < 0$, $f_{yy} < 0$, and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. The SOC's for a minimum are $f_{xx} > 0$, $f_{yy} > 0$, and $f_{xx}f_{yy} - (f_{xy})^2 > 0$ (yes the 3rd one does not change). If these second derivatives are equations, you will need to plug in the (x,y) pair you identified as the solution to the FOC's.

Example 1:
Maximize $w(y,z) = -30 + 150y + 90z - 6y^2 - 12z^2 - 2yz$

FOC's: $\qquad w_y(y,z) = 150 - 12y - 2z = 0$
$\qquad\qquad W_z(y,z) = 90 - 24z - 2y = 0$

Solve the simultaneous equations:
1. Solve the first FOC for z as a function of y
2. Substitute this function into the second FOC and solve for y
3. Substitute this value for y into z as a function of y from step 1 to find z

From the first FOC: $\quad 150 - 12y - 2z = 0$
$\qquad\qquad\qquad\quad 2z = 150 - 12y$
$\qquad\qquad\qquad\quad z = 75 - 6y$

Substitute this into the second FOC:
$\qquad\qquad 90 - 24z - 2y = 0$
$\qquad\qquad 90 - 24(75 - 6y) - 2y = 0$
$\qquad\qquad 90 - 1800 + 144y - 2y = 0$
$\qquad\qquad 142y = 1710$
$\qquad\qquad y = 1710/142 = 12.04$

Substitute this into the equation for z
$\qquad\qquad z = 75 - 6y$
$\qquad\qquad z = 75 - 2(12.04) = 2.76$

SOCs for maximum:
$\qquad\qquad w_{yy}(y,z) = -12 < 0$ (satisfied)
$\qquad\qquad w_{zz}(y,z) = -24 < 0$ (satisfied)
$\qquad\qquad w_{yz}(y,z) = -2 < 0$
$\qquad\qquad w_{yy}w_{zz} - w_{yz2} = (-12)(-24) - (-2)^2 = 288 - 4 = 284 > 0$ (satisfied)
$\qquad\qquad$ All three SOC's for a maximum are satisfied.

Example 2:
Find x and y to maximize $m(x,y) = -1 + 9x + 6y - 3x^2 - 3y^2 - 3xy$

FOC's $\qquad m_x = 9 - 6x - 3y = 0$
$\qquad\qquad m_y = 6 - 6y - 3x = 0$

Solve FOC1 (m_x) for y as a function of x:
$\qquad\qquad 9 - 6x - 3y = 0$
$\qquad\qquad 3y = 9 - 6x$
$\qquad\qquad y = 3 - 2x$

Substitute this into FOC2 (m_y) and solve for x:
$$6 - 6y - 3x = 0$$
$$6 - 6(3 - 2x) - 3x = 0$$
$$6 - 18 + 12x - 3x = 0$$
$$6 - 18 + 12x - 3x = 0$$
$$9x = 12$$
$$x = 12/9 = 4/3$$

Substitute this value in the equation for y as a function x and solve for y:
$$y = 3 - 2x$$
$$y = 3 - 2(4/3) = 3 - 8/3 = 1/3$$

SOC's: $m_{xx} = -6 < 0$ (satisfied)
$m_{yy} = -6 < 0$ (satisfied)
$m_{xy} = -3$
$m_{xx}m_{yy} - m_{xy}^2 = (-6)(-6) - (-3)2 = 36 - 9 = 27 > 0$ satisfied)

The SOC's are satisfied for a maximum, so x = 4/3, y = 1/3 is a maximum.